Appendage Modal Coordinate Truncation Criteria in Hybrid Coordinate Dynamic Analysis

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Seven alternative candidates for hybrid and normal coordinate truncation criteria are offered for consideration, interpreted, and illustrated by example. Two of the criteria are based on system eigenvalues, two on system eigenvectors, and three on measures of controllability and observability. No definitive basis is provided for selection among the seven criteria, but comparisons are offered.

Introduction

THE practice of hybrid coordinate dynamic analysis for attitude control of spacecraft with flexible appendages[1] is now well established in engineering practice, despite the fact that certain of the essential assumptions of this approach have never been formally justified, or even critically examined. It is the purpose of this paper to confront the question of the validity of the assumption that certain appendage distributed (modal) coordinates can be truncated from a system model without unacceptable corruption of the fidelity of the dynamical simulation. Alternative truncation criteria are proposed and their interrelationships defined, but no definitive resolution of the problem is advanced in this paper; nor indeed does it seem that this issue can be wholly resolved in terms that are mathematically rigorous and still useful to practicing engineers.

Background

Classical vibration theory[2] tells us that given a system of n linear, constant coefficient equations in the matrix form

$$M\ddot{q}+C\dot{q}+Kq=F(t) \tag{1}$$

with symmetric M and K, and with C a linear combination of M and K, there exists a transformation matrix Φ such that $q=\Phi w$ provides (after premultiplication by Φ^T) a system of n independent scalar equations, which with suitable definitions may be recorded as the matrix equation

$$\ddot{w}+2\xi p\dot{w}+p^2 w=Q(t) \tag{2}$$

with diagonal coefficient matrices p and ξ comprised of normal mode natural frequencies and damping ratios, respectively. Obviously each of the uncoupled coordinates $w_i(t)$ $(i=1,\ldots,n)$ is a function of time which can be determined independently of all others, so that if instead of Eq. (2) we consider the truncated version

$$\ddot{\bar{w}}+2\bar{\xi}\bar{p}\dot{\bar{w}}+\bar{p}^2\bar{w}=\bar{Q}(t) \tag{3}$$

with $\bar{w}$ of dimension $\bar{n}\times 1$, with $\bar{n}<n$, and with other barred matrices representing truncated forms of the corresponding unbarred matrices, then the coordinates preserved after truncation are unaltered by the truncation process. If now one partitions the transformation to obtain

$$q=\Phi w=[\bar{\Phi}\bar{\bar{\Phi}}]\begin{Bmatrix}\bar{w}\\ \hline \bar{\bar{w}}\end{Bmatrix}=\bar{\Phi}\bar{w}+\bar{\bar{\Phi}}\bar{\bar{w}} \tag{4}$$

and then chooses to replace q by

$$\bar{q}=\bar{\Phi}\bar{w} \tag{5}$$

he is clearly introducing an *approximation* for q which cannot be *formally* justified. This is nonetheless a ubiquitous practice, and certainly it can often provide a justifiable approximation by practical standards of engineering analysis. The analyst is secure in the knowledge that he can always evaluate the error in his approximation by simply examining (one by one) the additive contributions to $q(t)$ of the elements of $\bar{\bar{w}}(t)$. Obviously, the truncation procedure cannot be justified even in an engineering sense for an arbitrary transformation operator Φ; the fact that Eq. (3) describes *uncoupled* scalar equations provides an important argument in the practical justification of modal coordinate truncation.

In the hybrid coordinate approach,[1] the equations of primary rigid body rotation and appendage deformation have the form (after transformation to distributed coordinates for appendage deformations)

$$I\ddot{\theta}-\delta^T\ddot{\eta}=T \tag{6}$$

$$\ddot{\eta}+2\zeta\sigma\dot{\eta}+\sigma^2\eta-\delta\ddot{\theta}=0 \tag{7}$$

(See Ref. 1, Eq. 288.)§ Here θ is a 3×1 matrix of central body attitude angles, η is an $N\times 1$ matrix of appendage modal coordinates, and T is a 3×1 matrix of torque components. The matrix δ is called the coupling matrix, since it couples together

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§In the hybrid coordinate approach, rigid body attitude angles are not always linearized, but when, as in this paper, the entire system of equations is linearized the result is no more than a special case of *component modal synthesis*, as introduced by Hurty.[3]

rotations and deformations. In application these equations are truncated to obtain

$$I\ddot{\theta}-\bar{\delta}^T\ddot{\bar{\eta}}=T \tag{8}$$

$$\ddot{\bar{\eta}}+2\bar{\zeta}\bar{\sigma}\dot{\bar{\eta}}+\bar{\sigma}^2\bar{\eta}-\bar{\delta}\ddot{\theta}=0 \tag{9}$$

where $\bar{\eta}$ has dimension $\bar{N}\times 1$, with $\bar{N}<N$. While the coefficient matrices in Eqs. (7) and (9) are diagonal, indicating that if $\ddot{\theta}$ is constrained to be zero then the appendage modal vibrations are uncoupled *normal modes* of vibration, for the given (unconstrained) boundary conditions the modal coordinates in η and $\bar{\eta}$ are *not* normal mode coordinates for the system. Yet we find in numerous papers and in common engineering practice the routine application of coordinate truncation to appendage modes in hybrid coordinate formulations of spacecraft attitude control problems. We might reasonably ask for more justification than has heretofore been advanced, and seek to develop new criteria for truncation if we do perceive the practice to have merit under certain conditions. In what follows, we examine alternative criteria, with the understanding that our objective is not necessarily to formally justify the truncation procedure, but in some cases merely to achieve, if possible, the same degree of justification for hybrid coordinate truncation that is available for the classical normal mode approach.

Truncation Criteria Based on Eigenvalues

In the classical normal mode theory, the most commonly advanced criterion for truncation is based on the natural frequencies of the total system, as represented by p_j $(j=1,\ldots,n)$ in Eq. (2), or more generally by the eigenvalues of the system, as represented by $-\xi_j p_j \pm i p_j(1-\xi_j^2)^{1/2}$ $(j=1,\ldots,n)$ for the system in Eq. (2). Typically, modes having the lowest frequency are retained after truncation, and any node η_j which exhibits resonance or near-resonance with the forcing function $Q_j(t)$ is also preserved. Thus we have what we will call truncation criterion 1, as follows:

Criterion 1. Normal mode coordinates can be truncated if their frequencies substantially exceed that of the highest significant harmonic in the forcing function; for free vibrations, normal mode coordinates of lowest frequency should be preserved in truncation.

While the merits of this classical truncation criterion can be defended analytically in some cases and disputed in others, we set aside this old question here to examine a candidate truncation criterion for hybrid coordinate formulations which is implied by the acceptance of the traditional criterion 1 for normal coordinate formulations.

Criterion 2. Appendage modal coordinates in hybrid coordinate formulations can be truncated if the eigenvalues of the truncated hybrid coordinate equations [Eqs. (8) and (9)] provide a "good" approximation of the eigenvalues of the original system [Eqs. (6) and (7)] which would be preserved in a normal coordinate truncation by criterion 1.

In order to make this criterion useful, we must find a way of relating these two sets of eigenvalues. This objective is greatly facilitated by the results of Ref. 4.

For the special (but typical) case of small modal damping, we can ignore ζ in Eq. (7) and $\bar{\zeta}$ in Eq. (9) and substantially simplify the eigenvalue (or the natural frequency) comparisons. Thus we can replace Eqs. (6-9) with undamped and unforced versions, and for analytical convenience also replace θ by the new attitude variable ψ and introduce the new symbol Δ:

$$\psi \triangleq I^{1/2}\theta \tag{10}$$

$$\Delta \triangleq \delta I^{-1/2} \tag{11}$$

to obtain for this comparison

$$\ddot{\psi}-\Delta^T\ddot{\eta}=0 \tag{12a}$$

$$\ddot{\eta}+\sigma^2\eta-\Delta\ddot{\psi}=0 \tag{12b}$$

and

$$\ddot{\psi}-\bar{\Delta}^T\ddot{\bar{\eta}}=0 \tag{13a}$$

$$\ddot{\bar{\eta}}+\bar{\sigma}^2\bar{\eta}-\bar{\Delta}\ddot{\psi}=0 \tag{13b}$$

If now $\lambda_1',\ldots,\lambda_{2\bar{N}}'$ are the nonzero eigenvalues of Eq. (13), and $\lambda_1,\ldots,\lambda_{2\bar{N}}$ are the $2\bar{N}$ nonzero eigenvalues of Eq. (12) to be preserved after truncation (often those with smallest modulus), then the challenge is to obtain expressions for $\lambda_1',\ldots,\lambda_{2\bar{N}}'$ in terms of $\lambda_1,\ldots,\lambda_{2\bar{N}}$. Of course both Eqs. (12) and (13) have six zero eigenvalues.

If we define μ_j and μ_j' by

$$\mu_j \triangleq -(\lambda_j)^2 \quad j=1,\ldots,2N \tag{14a}$$

and

$$\mu_j' \triangleq -(\lambda_j')^2 \quad j=1,\ldots,2\bar{N} \tag{14b}$$

then the problem is to compare μ_j and μ_j' where μ_j' is a root of

$$|\mu U_{\bar{N}}-\bar{M}_I\bar{\sigma}^2|=0 \tag{15}$$

with¶

$$\bar{M}_I \triangleq (U_{\bar{N}}-\bar{\Delta}\bar{\Delta}^T)^{-1} \tag{16}$$

and μ_j is a root of

$$|\mu U_N-M_I\sigma^2|=0$$

with

$$M_I \triangleq (U_N-\Delta\Delta^T)^{-1}$$

For convenience of later discussions, we also define $\bar{\bar{\Delta}}$, $\bar{\bar{\sigma}}$, and $\bar{\bar{M}}_I$ for the part of the system to be deleted by truncation:

$$\bar{\bar{\Delta}} \triangleq \begin{bmatrix} \Delta^{\bar{N}+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Delta^N \end{bmatrix} \tag{17}$$

$$\bar{\bar{\sigma}} \triangleq \begin{bmatrix} \sigma_{\bar{N}+1} & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \sigma_N \end{bmatrix} \tag{18}$$

and

$$\bar{\bar{M}}_I \triangleq \left(U_{\bar{\bar{N}}}-\bar{\bar{\Delta}}\bar{\bar{\Delta}}^T\right)^{-1} \tag{19}$$

where

$$\bar{\bar{N}} \triangleq N-\bar{N} \tag{20}$$

The notation μ_j' will be used also for the roots of

$$|\mu U_{\bar{\bar{N}}}-\bar{\bar{M}}_I\sigma^2|=0 \tag{21}$$

¶M_I and $\bar{M}_I$ are shown to exist in Ref. 4.

with

$$j=\bar{N}+1,\bar{N}+2,\ldots,N.$$

With these definitions, we may write

$$\sigma=\left[\begin{array}{c|c}\bar{\sigma} & 0\\ \hline 0 & \bar{\bar{\sigma}}\end{array}\right] \tag{22}$$

and

$$\Delta=\left[\begin{array}{c}\bar{\Delta}\\ \hline \bar{\bar{\Delta}}\end{array}\right] \tag{23}$$

In what follows, we will examine the reciprocals of the roots defined by

$$\nu_j \triangleq \frac{1}{\mu_j} \qquad (j=1,2,\ldots,N)$$

which satisfy

$$|\nu U_N-\sigma^{-1}M_I\sigma^{-1}|=0$$

and by

$$\nu_j' \triangleq \frac{1}{\mu_j'} \tag{24}$$

which satisfy (as in Eq. (51) of Ref. 4)

$$|\nu' U_N-\bar{\sigma}^{-1}\bar{M}_I^{-1}\bar{\sigma}^{-1}|=0 \tag{25}$$

for

$$j=1,2,\ldots,\bar{N}$$

and

$$|\nu' U_{\bar{\bar{N}}}-\bar{\bar{\sigma}}^{-1}\bar{\bar{M}}_I^{-1}\bar{\bar{\sigma}}^{-1}|=0 \tag{26}$$

for

$$j=\bar{N}+1,\ \bar{N}+2,\ldots,N$$

Since M_I^{-1} is partitioned as

$$M_I^{-1}=\left[\begin{array}{c|c}U_{\bar{N}} & 0\\ \hline 0 & U_{\bar{\bar{N}}}\end{array}\right]-\left[\begin{array}{c}\bar{\Delta}\\ \hline \bar{\bar{\Delta}}\end{array}\right]\left[\begin{array}{c}\bar{\Delta}\\ \hline \bar{\bar{\Delta}}\end{array}\right]^T$$

$$=\left[\begin{array}{c|c}U_{\bar{N}}-\bar{\Delta}\bar{\Delta}^T & -\bar{\Delta}\bar{\bar{\Delta}}^T\\ \hline -\bar{\bar{\Delta}}\bar{\Delta}^T & U_{\bar{\bar{N}}}-\bar{\bar{\Delta}}\bar{\bar{\Delta}}^T\end{array}\right]$$

$$=\left[\begin{array}{c|c}\bar{M}_I^{-1} & 0\\ \hline 0 & \bar{\bar{M}}_I^{-1}\end{array}\right]+\left[\begin{array}{c|c}0 & -\bar{\Delta}\bar{\bar{\Delta}}^T\\ \hline -\bar{\bar{\Delta}}\bar{\Delta}^T & 0\end{array}\right]$$

the matrix $\sigma^{-1}M_I^{-1}\sigma^{-1}$ is decomposed and rewritten as

$$\sigma^{-1}M_I{}^{-1}\sigma^{-1}=\mathfrak{D}+\mathfrak{F}$$

where

$$\mathfrak{D}\triangleq\left[\begin{array}{c|c}\bar{\sigma}^{-1}\bar{M}_I^{-1}\bar{\sigma}^{-1} & 0\\ \hline 0 & \bar{\bar{\sigma}}^{-1}\bar{\bar{M}}_I^{-1}\bar{\bar{\sigma}}^{-1}\end{array}\right] \tag{27}$$

and

$$\mathfrak{F}\triangleq\left[\begin{array}{c|c}0 & -\bar{\sigma}^{-1}\bar{\Delta}\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-1}\\ \hline -\bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\bar{\Delta}^T\bar{\sigma}^{-1} & 0\end{array}\right] \tag{28}$$

It is recognized that the eigenvalues of $\mathfrak{D}$ are ν_j', which are determined by Eqs. (25) and (26), because

$$|\nu U_N-\mathfrak{D}|=|\nu U_{\bar{N}}-\bar{\sigma}^{-1}\bar{M}_I^{-1}\bar{\sigma}^{-1}|\cdot|\nu U_{\bar{\bar{N}}}-\bar{\bar{\sigma}}^{-1}\bar{\bar{M}}_I^{-1}\bar{\bar{\sigma}}^{-1}| \tag{29}$$

From Eq. (28), if $\bar{\Delta}\bar{\bar{\Delta}}^T=0$ then $\mathfrak{F}=0$ and

$$\nu_j=\nu_j'$$

and hence

$$\mu_j=\mu_j' \tag{30}$$

In other words, if $\Delta^j\Delta^{kT}=0$ for $j=1,2,\ldots,\bar{N}$ and $k=\bar{N}+1,\ldots,N$ then the eigenvalues of the truncated system together with those of the deleted system are identical to the eigenvalues of the original system.

If $\bar{\Delta}\bar{\bar{\Delta}}^T\neq 0$, then the matrix $\mathfrak{F}$ usually causes a discrepancy between ν_j and ν_j'. In order to examine this influence, first consider the eigenvalues of $\mathfrak{F}$, to be called ω_j. The ω_j are the roots of

$$|\omega U_N-\mathfrak{F}|=0$$

or from Eq. (29)

$$\left|\begin{array}{c|c}\omega U_{\bar{N}} & -\bar{\sigma}^{-1}\bar{\Delta}\bar{\bar{\Delta}}\bar{\bar{\sigma}}^{-1}\\ \hline -\bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\bar{\Delta}^T\bar{\sigma}^{-1} & \omega U_{\bar{\bar{N}}}\end{array}\right|=0 \tag{31}$$

For nonzero ω, the determinant identity applies to yield

$$|\omega U_{\bar{\bar{N}}}-\bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\bar{\Delta}^T\bar{\sigma}^{-1}(\omega U_{\bar{N}})^{-1}\bar{\sigma}^{-1}\bar{\Delta}\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-1}|=0$$

or

$$|\omega^2 U_{\bar{\bar{N}}}-\bar{\bar{\sigma}}^{-1}\bar{\bar{\Delta}}\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta}\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-1}|=0 \tag{32a}$$

or

$$|\omega^2 U_3-(\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})|=0 \tag{32b}$$

It turns out from Eq. (32) that there exist at most six nonzero eigenvalues of $\mathfrak{F}$. Moreover, since $\mathfrak{F}$ is symmetric, all the eigenvalues are real, so that if we call the roots of Eq. (32) $\pm\omega_1^0$, $\pm\omega_2^0$, $\pm\omega_3^0$ with $\omega_1^0\geq\omega_2^0\geq\omega_3^0\geq 0$, then the ω_j's are given in nonincreasing order by

$$\omega_1^0,\ \omega_2^0,\ \omega_3^0,\ \underbrace{0,\ 0,\ldots,0}_{(N-6)\text{ zeros}}-\omega_3^0,\ -\omega_2^0,\ -\omega_1^0$$

From the fact that the matrix $\sigma^{-1}M_I{}^{-1}\sigma^{-1}$ is a sum of two matrices, $\mathfrak{D}$ and $\mathfrak{F}$, which are both symmetric, we may apply the Wielandt-Hoffman theorem,[5] stating that if ν_j and ν_j' and ω_j are arranged in nonincreasing (or nondecreasing) order, then

$$\sum_{j=1}^{N}(\nu_j-\nu_j')^2\leq\sum_{j=1}^{N}\omega_j^2 \tag{33}$$

Noting that

$$\sum_{j=1}^{N}\omega_j^2=2\sum_{j=1}^{3}\omega_j^2=2\sum_{j=1}^{3}(\omega_j^0)^2=2\ \mathrm{tr}(\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})$$

and recalling the definition of Eq. (24), we see that Eq. (33) immediately implies that

$$\sum_{j=1}^{N}\left(\frac{1}{\mu_j}-\frac{1}{\mu_j'}\right)^2 \leq 2\mathrm{tr}(\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}}) \tag{34}$$

Equation (34) gives a bound of errors which could take place if we approximated μ_j by μ_j'. If the relative errors are small, i.e.,

$$|\epsilon_\ell/\mu_\ell'|<<1$$

where

$$\epsilon_\ell \triangleq \mu_\ell'-\mu_\ell$$

then

$$\frac{1}{\mu_\ell}-\frac{1}{\mu_\ell'}=\frac{1}{\mu_\ell'-\epsilon_\ell}-\frac{1}{\mu_\ell'}=\frac{1}{\mu_\ell'}\cdot\frac{\epsilon_\ell}{\mu_\ell'}+O(\epsilon_\ell^2)$$

Therefore,

$$\sum_{j=1}^{N}\left(\frac{1}{\mu_j}-\frac{1}{\mu_j'}\right)^2=\sum_{\ell=1}^{N}\left(\frac{1}{\mu_\ell'}\right)^2\left(\frac{\epsilon_\ell}{\mu_\ell'}\right)^2+O(\epsilon_\ell^4)$$

and if we neglect $0(\epsilon_\ell^4)$, then

$$\sum_{\ell=1}^{N}\left(\frac{1}{\mu_\ell'}\right)^2\left(\frac{\epsilon_\ell}{\mu_\ell'}\right)^2 \leq 2\mathrm{tr}(\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})$$

Each term in the summation of the left-hand side is positive, so that for any ℓ

$$\left(\frac{1}{\mu_\ell'}\right)^2\left(\frac{\epsilon_\ell}{\mu_\ell'}\right)^2 \sim \mathrm{tr}(\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})$$

or

$$\left|\frac{\epsilon_\ell}{\mu_\ell'}\right| \leq \mu_\ell'[2\mathrm{tr}(\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}})]^{1/2} \quad \ell=1,2,...,N \tag{35}$$

Equation (35) gives a bound for the relative errors, and if the dimensionless quantity of the right-hand side of Eq. (35) is sufficiently small, judging from some practical point of view, then we may employ μ_j' as a satisfactory approximation of μ_j, permitting the truncation to be acceptable. This condition, however, is a sufficient one in that even if the quantity in question is not small enough this does not necessarily imply that the μ_j' are unacceptable. This is because the error limits of Eq. (35) are overestimated due to the neglect of the positive terms in the left-hand side of Eq. (35).

Equation (34) also implies that

$$\sum_{j=1}^{\bar{N}}\left(\frac{1}{\mu_j}-\frac{1}{\mu_j'}\right)^2 \leq 2\mathrm{tr}(\bar{\Delta}^T\bar{\sigma}^{-2}\bar{\Delta})(\bar{\bar{\Delta}}^T\bar{\bar{\sigma}}^{-2}\bar{\bar{\Delta}}) \tag{36}$$

The requirement that the quantity of the right-hand side of Eq. (36) is practically small is also a sufficient condition in the sense stated previously.

The minimax theorem[4] as applied to the matrices $\sigma^{-1}M_I^{-1}\sigma^{-1}$, $\mathfrak{D}$, and $\mathfrak{F}$ produces the following result. Since the eigenvalues of $\sigma^{-1}M_I^{-1}\sigma^{-1}$, $\mathfrak{D}$, and $\mathfrak{F}$ are ν_j and ν_j', respectively, and the maximum and minimum eigenvalues of $\mathfrak{F}$ are ω_1^0 and $-\omega_1^0$, we have

$$\nu_j'-\omega_1^0 \leq \nu_j \leq \nu_j'+\omega_1^0 \tag{37}$$

or

$$\frac{\mu_j'}{1+\mu_j'\omega_1^0} \leq \mu_j \leq \frac{\mu_j'}{1-\mu_j'\omega_1^0} \tag{38}$$

Equation (38) affords an explicit error bound for μ_j' at the cost of solving the eigenvalue problem of $\mathfrak{F}$, which is a 3×3 symmetric matrix.

Equations (34-38) provide various measures of the relationship between the eigenvalues of the truncated system and the selected $2\bar{N}$ eigenvalues of the original system, thereby permitting application of the candidate truncation criteria 1 and 2.

Truncation Criteria Based on Eigenvectors

The hybrid coordinate approach to flexible spacecraft dynamic analysis is adopted only when the vehicle can reasonably be idealized as a rigid body with flexible appendages. In such a case one is frequently concerned not with the whole range of output variables but solely with θ or with θ and $\dot{\theta}$. It then becomes *formally* valid to truncate any modes which have no influence on $\theta(t)$. This truncation rationale is appropriate whether one employs vehicle normal coordinates or hybrid coordinates, so that we have the following truncation criteria:

Criterion 3. A normal coordinate can be deleted by truncation if in its eigenvector or mode shape function there is a zero in the entry of the vector space corresponding to the selected output variable θ.

Criterion 4. In a hybrid coordinate formulation, appendage modal coordinates can be deleted by truncation if the corresponding row in the coupling matrix [δ in Eqs. (6) and (7)] is null, or its elements are sufficiently small.

The justification for criterion 4 is the same as that offered for criterion 3; a zero column in δ^T clearly indicates that one of the appendage modal vibration coordinates in η does not excite rotational accelerations $\ddot{\theta}$ of the base. This criterion is not new, having been noted at least in Refs. 1 and 6. It has been observed[1] that for the jth appendage mode the 3×3 matrix $\delta^{jT}\delta^j$ appears in the transfer function relating Laplace transforms of $\theta(t)$ and $T(t)$ as an "effective inertia matrix" for the jth mode, and this matrix is now routinely used in engineering practice to provide a truncation criterion. Since this matrix depends upon appendage mode shape and mass distribution, this truncation criterion is in the class of criteria based on eigenvectors.

Truncation Criteria Based on Controllability and Observability

The traditions of modal coordinate truncation are rooted in the field of structural dynamics, but the methods are applicable in other areas, such as model reduction in general systems analysis. Thus, it is perfectly natural when considering problems of flexible spacecraft attitude control to develop coordinate truncation criteria expressed in terms of the basic measures of control system performance, such as controllability, observability, stability, and various cost or performance indices. This subject has apparently not been explored, so we enter it here, if only in tentative fashion.

It is appropriate for what follows that we recast the basic equations of motion [see Eqs. (1-9)] in the state variable form

$$\dot{x}=AX+Bu \tag{39a}$$

where $x(t)$ is an n-dimensional state vector and $u(t)$ is an s-dimensional control vector. The output $y(t)$ and the measurement $z(t)$ are, respectively, r-dimensional and m-dimensional linear functions of $x(t)$, given by

$$y=Cx \tag{39b}$$

$$z=DX \tag{39c}$$

In what follows, we consider the special time-invariant case, with constant matrices A, B, C, and D.

The concepts of controllability and observability[7-9] are defined as follows for the system of Eq. (39). The system is *completely controllable* if there exists a bounded control $u(t)$ which can bring the state $x(t)$ in a finite time interval t_f from any initial value $x(0)$ to the origin, so that $x(t_f)=0$. The system is *completely observable* if knowledge of $z(t)$ and $u(t)$ for $0 \leq t \leq t_f$ permits the determination of $x(t_f)$.

Output controllability and output observability are defined as before, but with $y(t)$ substituted for $x(t)$. As necessary and sufficient conditions for complete controllability and complete observability, respectively,[7] there must exist nonzero values of the controllability determinant Δ_c and the observability determinant Δ_0, defined as follows for the system of Eq. (39):

$$\Delta_0 \triangleq |Q_c Q_c^T| \tag{40}$$

where

$$Q_c \triangleq [B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B] \tag{41}$$

and

$$\Delta_0 \triangleq |Q_0 Q_0^T| \tag{42}$$

where

$$Q_0 \triangleq [D^T \mid A^T D^T \mid (A^T)^2 D^T \mid \cdots \mid (A^T)^{n-1} D^T] \tag{43}$$

Continuing the citation of established results,[7-9] we can observe that for the fixed-time minimum energy transfer control problem, the minimum control energy is a quadratic form in the initial state

$$E = x(0)^T W_c^{-1}(t_f) x(0)$$

where for our problem

$$W_c(t_f) = \int_0^{t_f} e^{-A\tau} B B^T e^{-A^T\tau} \mathrm{d}\tau$$

$$= Q_c \left[\int_0^{t_f} \alpha(-\tau) \alpha^T(-\tau) \mathrm{d}\tau \right] Q_c^T$$

with the elements of α defined by

$$e^{A\tau} = \alpha_0(\tau) U + \alpha_1(\tau) A + \alpha_2(\tau) A^2 + \ldots + \alpha_{n-1}(\tau) A^{n-1}$$

(assuming that the degree of the minimal polynomial of A is n). Motivated by the fact that

$$E = x(0)^T \left\{ Q_c \left[\int_0^{t_f} \alpha(-\tau) \alpha^T(-\tau) \mathrm{d}\tau \right] Q_c^T \right\}^{-1} x(0)$$

we choose in what follows to attach significance to $(Q_c Q_c^T)^{-1}$, which has the advantage of independence of $x(0)$ and t_f. This interpretation, together with the physical significance of the concepts of controllability and observability, suggest further candidate truncation criteria:

Criterion 5. Coordinate truncation should be accomplished so as to minimally influence the trace of $(Q_c Q_c^T)^{-1}$.

Criterion 6. If the deletion of a single coordinate changes both Δ_c and Δ_0 from zero to nonzero values, and if output controllability and output observability are uninfluenced by this deletion, then this coordinate should be removed by truncation.

Criterion 7. For a given degree of truncation, coordinates should be preserved which maximize the ratio Δ_0 / Δ_c.

Motivation and justification for these proposed criteria can be argued tentatively as follows: Criterion 5 provides an approximate scalar measure of the minimum energy required for control. As noted in Ref. 8, other scalar measures might be considered (such as the maximum eigenvalue of $(Q_c Q_c^T)^{-1}$ or its determinant), but only the trace possesses the appropriate sensitivity to coordinate truncation selection. Criterion 6 mandates the elimination from the model of any modal coordinate that makes the system both not completely controllable and not completely observable, provided that output controllability and output observability are uninfluenced by that coordinate. The retention in the system model of such a coordinate serves no useful purpose, and increases the state dimension unnecessarily. Criterion 7 is the least well motivated of those listed here, but in a rough sense it embodies both criteria 5 and 6, and experience in applications reveals that it also has for the single appendage mode hybrid coordinate case a conceptual similarity to criterion 4.

Example

Figure 1 portrays an idealized spacecraft consisting of a rigid central body with a pair of symmetrically located identical elastic appendages. An attitude sensor and a torque actuator on the central body accomplish closed loop attitude control for planar motion.

The equations of motion in second order form appear as Eqs. (6) and (7), and the output and measurement equations [Eqs. (39b) and (39c)] reduce to the scalars

$$y = \theta, \quad z = \theta \tag{44}$$

If $\phi_j(\xi)$ is the jth modal function, with ξ a coordinate along the appendage length, then the jth entry in the $n \times 1$ coupling matrix δ is given by

$$\delta_j = -\frac{M}{L} \int_{-(L+d)}^{(L+d)} \phi_j(\xi) \xi \mathrm{d}\xi \tag{45}$$

where M is the mass and d and L are the dimensions shown in the figure.

By inspection it is apparent that appendage modal functions which are normal modes for θ constrained to zero must appear in two categories: symmetric (with $\phi(\xi) = \phi(-\xi)$, and asymmetric [with $\phi(\xi) = -\phi(-\xi)$]. Thus we have for all symmetric modes

$$\delta_j = 0 \quad j \text{ even} \tag{46a}$$

and for all asymmetric modes

$$\delta_j = -\frac{2M}{L} \int_d^{d+L} \phi_j(\xi) \xi \mathrm{d}\xi \quad j \text{ odd} \tag{46b}$$

When presented in state equation form, as in Eq. (39), we may choose, if three modes are retained for example,

$$x^T = [\theta \dot{\theta} \eta_i \dot{\eta}_i \eta_j \dot{\eta}_j \eta_k \dot{\eta}_k] \text{ and } u = T \tag{47a}$$

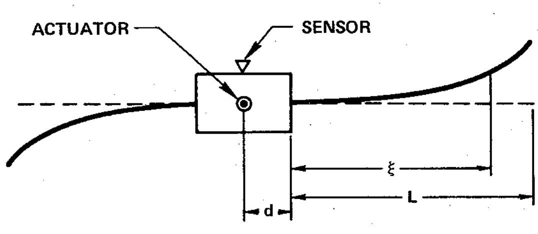

Fig. 1 Model of example.

and then

$$A=\frac{1}{J}\begin{bmatrix} 0 & J & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_i\sigma_i^2 & 0 & -\delta_j\sigma_j^2 & 0 & -\delta_k\sigma_k^2 & 0 \\ 0 & 0 & 0 & J & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma_i^2(J+\delta_i^2) & 0 & -\delta_i\delta_j\sigma_j^2 & 0 & -\delta_i\delta_k\sigma_k^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & J & 0 & 0 \\ 0 & 0 & -\delta_i\delta_j\sigma_i^2 & 0 & -\sigma_j^2(J+\delta_j^2) & 0 & -\delta_j\delta_k\sigma_k^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J \\ 0 & 0 & -\delta_j\delta_k\sigma_i^2 & 0 & -\delta_j\delta_k\sigma_j^2 & 0 & -\sigma_k^2(J+\delta_k^2) & 0 \end{bmatrix} \tag{47b}$$

and

$$B^T=\frac{1}{J}\,[0\ 1\ 0\ \delta_i\ 0\ \delta_j\ 0\ \delta_k] \tag{47c}$$

Here J is the reduced effective inertia of the vehicle

$$J=I-\delta_i^2-\delta_j^2-\delta_k^2 \tag{47d}$$

which is the actual moment of inertia of the total vehicle less the sum of the effective inertias of the three modes. As shown in Refs. 1 and 6, $J>0$. From these explicit equations for $N=3$, one can readily construct the corresponding equations for other values of N. Subsequent calculations are presented only for $N\leq 2$, for convenience of presentation.

In order to illustrate the preceding truncation criteria in application to this hybrid coordinate example, we will treat explicitly only the academic case in which we begin with $N=2$ (including a symmetric mode and an asymmetric mode), and consider first the truncation to $\bar{N}=1$, and then the truncation to $\bar{N}=0$. We present results only, with a minimum of development.

Before we can entertain the classical normal mode frequency criterion (1), we must apply criterion 2 and compare the frequencies of the truncated hybrid coordinate system to those of the normal mode system of like order. In this case, the required analysis tells us that if we delete either the symmetric mode or the asymmetric mode in truncating from $N=2$ to $\bar{N}=1$, then criterion 2 is fully satisfied, since $\bar{\Delta}^T\bar{\Delta}=0$, and if we proceed further to $\bar{N}=0$, then we have no nonzero eigenvalues and again criterion 2 is met. Thus, for this simple case, we can justify hybrid coordinate truncation just as easily as normal mode truncation. Criterion 1 is, however, not very useful, since no obvious selection is indicated between the truncation of the symmetric and the asymmetric mode.

In criteria 3 and 4, however, we find a clear indication that the symmetric mode should be deleted by coordinate truncation. Criterion 4 is easiest to apply, since Eq. (46a) tells us immediately that the symmetric mode (with $\delta=0$) contributes nothing of dynamical significance to the attitude control model, so it should be deleted. If one troubled to find the system eigenvectors (see Ref. 4), it would become apparent that the frequency of appendage symmetric vibration provides a system eigenvalue exactly, and the corresponding eigenvector makes no contribution to the output θ; thus criterion 3 also mandates deletion of the symmetric mode.

For the application of criteria 5,6, and 7, we require for $N=2$

$$Q_cQ_c^T=\frac{1}{J^2}\begin{bmatrix} 1+a^2+\alpha^2 & 0 & \delta_i+ab+\alpha\beta & 0 & \delta_j+ac+\alpha\gamma & 0 \\ 0 & 1+a^2+\alpha^2 & 0 & \delta_i+ab+\alpha\beta & 0 & \delta_j+ac+\alpha\gamma \\ \delta_j+ab+\alpha\beta & 0 & \delta_i^2+b^2+\beta^2 & 0 & \delta_i\delta_j+bc+\beta\gamma & 0 \\ 0 & \delta_i+ab+\alpha\beta & 0 & \delta_i^2+b^2+\beta^2 & 0 & \delta_i\delta_j+bc+\beta\gamma \\ \delta_j+ac+\alpha\gamma & 0 & \delta_i\delta_j+cb+\gamma\beta & 0 & \delta_j^2+c^2+\gamma^2 & 0 \\ 0 & \delta_j+ac+\alpha\gamma & 0 & \delta_i\delta_j+cb+\gamma\beta & 0 & \delta_j^2+c^2+\gamma^2 \end{bmatrix} \tag{48a}$$

$$Q_oQ_o^T=\frac{1}{J^4}\begin{bmatrix} J^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & J^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & J^2\delta_i^2\sigma_i^4+\alpha_o^2 & 0 & J^2\delta_i\delta_j\sigma_i^2\sigma_j^2+\alpha_o\beta_o & 0 \\ 0 & 0 & 0 & J^2\delta_i^2\sigma_i^4+\alpha_o^2 & 0 & J^2\delta_i\delta_j\sigma_j^2\sigma_j^2+\alpha_o\beta_o \\ 0 & 0 & J^2\delta_i\delta_j\sigma_i^2\sigma_j^2+\alpha_o\beta_o & 0 & J^2\delta_i^2\sigma_j^4+\beta_o^2 & 0 \\ 0 & 0 & 0 & J^2\delta_i\delta_j\sigma_i^2\sigma_j^2+\alpha_o\beta_o & 0 & J^2\delta_j^2\sigma_j^4+\beta_o^2 \end{bmatrix} \tag{48b}$$

where

$$\alpha \triangleq -\frac{1}{J}\left(\delta_i^2\sigma_i^2+\delta_j^2\sigma_j^2\right)$$

$$\beta \triangleq -\frac{1}{J}\left[\delta_i\sigma_i^2(J+\delta_i^2)+\delta_i\delta_j^2\sigma_j^2\right]$$

$$\gamma \triangleq -\frac{1}{J}\left[\delta_i^2\delta_j\sigma_i^2+\delta_j\sigma_j^2(J+\delta_j^2)\right]$$

$$a \triangleq -\frac{1}{J}\left[\delta_i\sigma_i^2\beta+\delta_j\sigma_j^2\gamma\right]$$

$$b \triangleq -\frac{1}{J}\left[\sigma_i^2\beta(J+\delta_i^2)+\delta_i\delta_j\sigma_j^2\gamma\right]$$

$$c \triangleq -\frac{1}{J}\left[\delta_i\delta_j\sigma_i^2\beta+\sigma_j^2\gamma(J+\delta_j^2)\right]$$

$$\alpha_0 \triangleq \delta_i\sigma_i^4(J+\delta_i^2)+\delta_i\delta_j^2\sigma_i^2\sigma_j^2$$

$$\beta_0 = \delta_i^2\delta_j\sigma_i^2\sigma_j^2+\delta_j\sigma_j^4(J+\delta_j^2) \tag{48c}$$

Criterion 5 requires the trace of the inverse of $Q_cQ_c^T$ for the given case with $N=2$, to compare with the corresponding traces when $\bar{N}=1$ and when $\bar{N}=0$. This calculation is simplified here by the fact that $\delta_i=0$ for the symmetric mode. Finally this criterion tells us that deleting the symmetric mode in truncation is always permissible, while deleting the asymmetric mode may not be.

Criterion 6 indicates that the symmetric mode must be deleted, since it renders both $Q_cQ_c^T$ and $Q_oQ_o^T$ singular, and the corresponding Δ_c and Δ_0 zero. This is an uncontrollable and unobservable mode for the given actuator and sensor, and it serves no useful purpose in the analysis.

Criterion 7 can be applied only after the symmetric mode is deleted, since with this mode in the model the indicated ratio is indeterminate. It is interesting, however, to note that for a single mode model we have (after laborious calculation)

$$\Delta_0 = \delta_j^4\sigma_j^8/J^4 \tag{49a}$$

and

$$\Delta_c = \delta_j^4\sigma_j^8/J^2 \tag{49b}$$

where now

$$J = I - \delta_j^2 \tag{49c}$$

so that

$$\Delta_0/\Delta_c = 1/J^2 \tag{50}$$

Criterion 7 tells us that the modal coordinate is important if it significantly influences the reduced effective inertia J of the vehicle. But from Eq. (49c), this is equivalent to the requirement that the mode be retained in truncation if its effective inertia δ_j^2 is significant in comparison with the vehicle moment of inertia I. Thus we see that criterion 7 is in this case a restatement of criterion 4 from a quite different perspective.

Conclusions

Seven candidate truncation criteria have been proposed for consideration, and each has received some interpretation and illustration. This is, however, only an entry to the subject, which requires much more exhaustive study before definitive truncation criteria can be advanced.

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